Hawk04-A Unitary Information Preservation and Radiation Recovery: A Complete-Complement Formulation of the Black-Hole Information Problem

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Abstract

We prove the precise conditional statement that quantum information cannot be destroyed by unitary evolution of a complete closed system. For a finite logical code, a fixed pure ancillary preparation, and a unitary evolution, the complete output channel has a CPTP left inverse, including every external reference. Every logical matrix unit and every input-state trace distance are preserved.

This global statement does not imply recovery from radiation alone. For a complete isometry $V: C \to R \otimes B$, we prove that one decoder recovers the entire code from R exactly if and only if the complete complementary channel to B is constant. A Schmidt-space construction provides the decoder and the exact no-hiding normal form. We distinguish this result from partial decoherence, restricted input ensembles, and mixed-output descriptions with omitted preparation purifiers.

For the regulated quartic SYK–scalar architecture of Hawk02-E, we reconstruct the self-adjoint dynamics and a reference-preserving finite approximation. A finite complementary Choi certificate e, input dimension d, and complete-channel regulator allowance b yield $\delta_{full} \le de + b$. A finite decoder can be lifted to the continuum with recovery error at most $\sqrt{(2de)} + b$, or certified directly using its decoded Choi operator. We also give an exact four-Majorana realization with nonzero coupling for which the optimal radiation recovery error is $(3 - z(t))/4 \ge 1/2$, although global recovery is exact. This is a dynamical negative control, not a scrambling argument.

Finally, we prove endpoint continuity with access-preserving output identifications and state the separate gravitational obligations. The results establish unitary preservation and operational recovery criteria. They do not establish unitarity of quantum gravity, a generic interacting evaporation-recovery window, or a gravitational endpoint.

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Central distinction.

$$\text{Global unitary preservation} \quad \not\Rightarrow \quad \text{recovery from a chosen radiation subsystem.}$$

The first assertion is proved from unitarity. The second requires a bound on the complete complementary channel. Assuming global unitarity does not prove that physical black-hole evaporation has that property.

Introduction, attribution, and scope

The assertion

``quantum information cannot be destroyed under unitary evolution''

is a theorem about reversible quantum dynamics. It is not, by itself, a derivation of the microscopic dynamics of an evaporating black hole.

Hawking's radiation calculation and his subsequent information-loss proposal provide the historical setting [Hawking1975,Hawking1976]. The relevant tension is between a complete reversible description and a proposed final description that fails to preserve arbitrary initial quantum states. Thermal occupation data, local radiation marginals, and a complete input-to-output quantum channel are different objects.

The exact no-hiding principle is due to Braunstein and Pati [BraunsteinPati]. Exact error correction and its syndrome structure are established in the work of Knill and Laflamme [KnillLaflamme]. Approximate information–disturbance relations are supplied by Kretschmann, Schlingemann, and Werner [KSW,KSWContinuity]. We do not claim these principles as new results. We give local proofs of the exact statements needed here and make the approximate external input explicit.

The supplied Cheung manuscripts motivate the complete-output discipline adopted in this paper. Hawk01-B distinguishes entropy histories from reference-preserving recovery. Hawk02-D supplies an autonomous quadratic benchmark with a complete scattering endpoint. Hawk02-E specifies an interacting regulated source–reservoir interface but does not establish its small dynamical residual bound [CheungHawk01B,CheungHawk02D,CheungHawk02E]. The operational manuscripts separately identify decoder execution, coherent collection, and channel-level error composition [CheungEn02B,CheungEn02C,CheungEn02E,CheungEm03C]. The dephasing comparisons in Em05-A and Em05-B distinguish delivered occupation from recovered quantum coherence [CheungEm05A,CheungEm05B].

Contribution. This paper supplies:

• a complete proof of global code preservation under unitary evolution;

• an exact complete-complement recovery theorem and decoder;

• counterexamples repairing overly broad readings of no-hiding;

• a self-contained finite-regulator certification interface for the specified interacting architecture;

• an exact, nonzero-coupling interacting-family negative control;

• an access-preserving endpoint continuity theorem. It is a theorem-and-certification paper, not a claim of a new universal resolution of black-hole evaporation.

Source policy. Cheung citations identify the supplied versions or excerpts. They are comparisons, not unprovided theorem premises. The model-specific arguments used below are proved in this source. No external mathematics catalogue is a premise.

Complete systems and operational definitions

Let $C \simeq \mathbb{C}^d$, with $d \geq 2$, be the logical code. Output Hilbert spaces are separable. States are normal density operators, and channels act on trace-class operators. CPTP means completely positive and trace preserving. Finite-dimensional channel facts and conventions are standard [Watrous].

We use

$$d_{\mathrm{tr}}(\rho,\sigma) = \frac{1}{2}\|\rho-\sigma\|_1, \qquad d_\diamond(\Phi,\Psi) = \frac{1}{2}\|\Phi-\Psi\|_\diamond.$$

The diamond distance retains arbitrary external references. For channels with input C, a reference of dimension d suffices.

For an isometry

$$V : C \to R \otimes B,$$

define

$$\mathcal{N}(X) = \mathrm{Tr}_B(VXV^\dagger), \qquad \mathcal{C}(X) = \mathrm{Tr}_R(VXV^\dagger).$$

Here B is the *complete* inaccessible output. It includes uncollected modes, retained source degrees of freedom, inaccessible apparatus outputs, and preparation purifiers.

For a residual state σ, set

$$\mathcal{T}_\sigma(X) = \mathrm{Tr}(X)\sigma.$$

Define

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}) = \inf_{\mathcal{D}\ \mathrm{CPTP}} d_\diamond(\mathcal{D}\mathcal{N}, \mathrm{id}_C),$$
$$\delta_{\mathrm{full}}(\mathcal{C}) = \inf_{\sigma} d_\diamond(\mathcal{C}, \mathcal{T}_\sigma).$$

Definition. A channel $\mathcal{V}: C \to \mathcal{H}_out$ preserves the code globally if a single CPTP map $\mathcal{L}$ satisfies

$$\mathcal{L}\mathcal{V} = \mathrm{id}_C.$$

This equality includes every code-reference input. It does not assert access to, or efficient implementation on, the complete output.

Definition. For a declared partition $R \otimes B$, radiation recovery means a single CPTP decoder $\mathcal{D}: R \to C$ satisfying

$$\mathcal{D}\mathcal{N} = \mathrm{id}_C$$

exactly, or the corresponding approximate diamond bound. The decoder is not allowed to act on B.

Mixed preparations. If an ancillary state is mixed, choose a fixed purification and retain its purifying system in the complete dilation. A code-dependent initial environment is a different task: the code-to-total-state preparation must then itself be specified and tested for preservation.

Logical coherences. With $E_ij = |i\rangle\langle j|$, the complete channel is determined by

$$\mathcal{V}(E_{ij}) = |Vi\rangle\langle Vj|.$$

Agreement on E_ii alone is not agreement on a quantum channel. For a qubit, both E_01 and $E_10 = E_01\^\{\dagger\}$ are required.

The unitary preservation theorem

Theorem. Let $J: C \to \mathcal{H}_in$ be an isometry, let $|\Omega\rangle$ be a fixed normalized ancillary state, and let U be unitary on the complete physical input space. After any fixed unitary regrouping of the output, set

$$V|\psi\rangle = U(J|\psi\rangle \otimes |\Omega\rangle).$$

Then:

• $V\^\{\dagger\}V = I_C$;

• the complete channel $\mathcal{V}(X) = VXV\^\{\dagger\}$ has a CPTP left inverse;

• for every trace-class operator X, including operators on the code and an arbitrary reference,

$$\|(V \otimes I)X(V^\dagger \otimes I)\|_1 = \|X\|_1;$$

• every logical coherence and every code-state trace distance are preserved;

• for a Bell input, the reference and the complete output have mutual information $2\log_2 d$. These assertions are conditional on the stated unitary dynamics and complete output space.

Proof. The composition of an isometry, a normalized pure preparation, and a unitary is an isometry, proving the first item.

Let $P = VV\^\{\dagger\}$, and choose any code density operator τ. Define

$$\mathcal{L}(Y) = V^\dagger Y V + \mathrm{Tr}((I - P)Y)\tau.$$

Both terms are completely positive. Their traces sum to $\mathrm{Tr}\, Y$. Thus $\mathcal{L}$ is CPTP, and

$$\mathcal{L}(VXV^\dagger) = X.$$

An isometric embedding leaves the nonzero singular values of an operator unchanged. Equivalently, its modulus is the isometric embedding of the original modulus on the range, with zero on the orthogonal complement. Taking its trace proves the third item, also after tensoring with a reference. Applying this identity to differences of density operators proves trace-distance preservation. Applying the defining isometry to *E_ij* proves preservation of all matrix units.

Finally,

$$|\Psi\rangle_{Q,\mathrm{out}} = (\mathrm{id}_Q \otimes V)|\Phi_d\rangle, \qquad |\Phi_d\rangle = d^{-1/2}\sum_i |i\rangle_Q|i\rangle_C,$$

is pure, and both reduced states have nonzero eigenvalues *1/d*. Therefore

$$I(Q:\mathrm{out}) = S(Q) + S(\mathrm{out}) - S(Q,\mathrm{out}) = 2\log_2 d.$$

All three entropies in this equality are finite, even if the complete output space is infinite dimensional.

Corollary. Distinct code states cannot have identical complete outputs under the hypotheses of Theorem . In particular, a complete replacement channel cannot describe that evolution on a nontrivial code.

Proof. Apply the CPTP left inverse, or use preserved trace distance.

Remark. The theorem concerns reversible encoding of a quantum state. It is not a conserved scalar particle number or a thermodynamic entropy-production law. Subsystem entropy may increase after a partial trace while the complete evolution remains reversible.

If *U(t) = e^{−itH}* for a self-adjoint Hamiltonian, the theorem applies at every finite time. The existence of that Hamiltonian is a separate analytical obligation. It is established for the regulated microscopic model in Section .

Exact recovery and the precise no-hiding statement

A complete-complement theorem

Theorem. Let $V: C \to R \otimes B$ be an isometry. The following are equivalent:

• a CPTP decoder on R exactly recovers the code;

• $\mathscr{C} = \mathscr{T}_{\sigma}$ for some residual density operator σ;

• for a code basis,

$$\mathscr{C}(E_{ij}) = \delta_{ij}\sigma;$$

• for the Bell input,

$$(\mathrm{id}_Q \otimes \mathscr{C})(|\Phi_d\rangle\langle\Phi_d|) = \frac{I_Q}{d} \otimes \sigma.$$

When these statements hold, there is an auxiliary Hilbert space K, a state σ_K, and an isometry $A: C \otimes K \to R$ such that

$$\mathscr{N}(X) = A(X \otimes \sigma_K)A^\dagger.$$

The decoder can be chosen as

$$\mathscr{D}(Y) = \mathrm{Tr}_K(A^\dagger Y A) + \mathrm{Tr}\,((I - AA^\dagger)Y)\frac{I_C}{d}.$$

Proof. The equivalence of (ii) and (iii) follows from linearity. Expanding the normalized Bell state in matrix units proves the equivalence of (iii) and (iv).

Assume (ii). Write the positive spectral decomposition

$$\sigma = \sum_\alpha p_\alpha |b_\alpha\rangle\langle b_\alpha|, \qquad p_\alpha > 0, \quad \sum_\alpha p_\alpha = 1.$$

The sum is finite or countable. Each code basis vector has an expansion

$$V|i\rangle = \sum_\alpha \sqrt{p_\alpha}\,|r_{i\alpha}\rangle_R|b_\alpha\rangle_B.$$

The equality of all complementary matrix units gives

$$\langle r_{j\beta}, r_{i\alpha}\rangle = \delta_{ij}\delta_{\alpha\beta}.$$

Thus the rule

$$A(|i\rangle \otimes |\alpha\rangle) = |r_{i\alpha}\rangle$$

extends to an isometry. Set $\sigma_K = \sum_\{\alpha\} p_\{\alpha\}|\alpha\rangle\langle\alpha|$. Tracing B gives .

The first term of is completely positive. The second is a positive-effect measurement followed by preparation. Their traces add to $Tr\ Y$, so the map is CPTP. Substitution into proves exact recovery.

Conversely, suppose (i). Use the normal Stinespring representation of the decoder, with an isometry

$$L : R \to C \otimes G.$$

The complete decoded isometry is

$$Z = (L \otimes I_B)V : C \to C \otimes G \otimes B.$$

Since its C output is exactly $|i\rangle$ on input $|i\rangle$,

$$Z|i\rangle = |i\rangle \otimes |\eta_i\rangle$$

for normalized η_i. The decoded off-diagonal matrix unit E_ij must also be exact. Tracing $G \otimes B$ therefore gives

$$\langle \eta_j, \eta_i\rangle = 1.$$

Hence all η_i are one common vector η. Tracing $C \otimes G$ shows

$$\mathscr{C}(E_{ij}) = \delta_{ij}\mathrm{Tr}_G\, |\eta\rangle\langle\eta|.$$

This proves (ii).

The theorem is the complete-dilation error-correction/no-hiding structure associated with [BraunsteinPati,KnillLaflamme]. The proof above is included to make the required output ledger and decoder explicit. Stinespring's representation is an external standard mathematical input [KSWContinuity,Watrous].

Corollary. If the accessible channel $\mathscr{N}$ is constant on the entire code, then the complete complementary output B exactly recovers that code. The arbitrary quantum state cannot reside only in inaccessible correlations between R and B.

Proof. Interchange R and B in Theorem .

Corollary. For $d \geq 2$, the two complementary channels of an isometry cannot both be constant on the entire code. Nor can both outputs separately recover the entire unknown code.

Proof. If one is constant, the other is exactly recoverable. A constant channel cannot recover two distinct inputs. If one output is exactly recoverable, its complement is constant by Theorem , and therefore is not itself recoverable.

The scope of ``not hidden in correlations''

Corollary requires:

• a complete isometric dilation;

• independence of the entire output channel on every code state;

• retention of all preparation purifiers. It does not say that correlations are irrelevant to recovery. It does not apply merely because one mean, one marginal for one input, or one selected input ensemble is unchanged.

Proposition. If the complete endpoint residual is a fixed pure state, then the radiation channel is an isometric encoding:

$$V_{\mathrm{end}}|\psi\rangle = A|\psi\rangle \otimes |\omega\rangle_B.$$

In particular, a pure code input has a pure *complete radiation* output.

If the fixed residual is mixed, radiation recovery is still exact, but a pure code input can have mixed radiation of the form .

Proof. In Theorem , a rank-one residual has a one-dimensional K. The general mixed case retains the input-independent syndrome state σ_K.

A horizonless effective geometry, by itself, establishes neither hypothesis of this proposition.

Reset, dephasing, and omitted purifiers

Reset by transfer

Example. Let S and E both contain a d-dimensional register, and prepare E in $|0\rangle$. The swap unitary gives

$$|\psi\rangle_S|0\rangle_E \text{longmapsto } |0\rangle_S|\psi\rangle_E.$$

The local channel on S is a reset, but the environmental channel is the identity on the input state. An entangled diary obeys the same identity with I_Q tensored onto the swap.

The example proves a transfer mechanism, not a universal account of practical reset hardware. A measurement record containing only a classical outcome does not, on its own, store an arbitrary qubit's phase.

Partial decoherence is not complete bleaching

Proposition. The isometry

$$V|0\rangle = |0\rangle_R|0\rangle_B, \qquad V|1\rangle = |1\rangle_R|1\rangle_B$$

has both reduced channels equal to computational dephasing:

$$\Delta_Z(\rho) = \text{diag}\,(\rho).$$

Neither output alone recovers an arbitrary qubit. Its optimal local recovery error is $1/2$, whereas the complete output is exactly recoverable.

Proof. Tracing either output annihilates E_{01} and E_{10}. The two inputs $|+\rangle$ and $|-\rangle$ have identical dephased outputs but orthogonal ideal targets. If a decoder has error ε, the triangle inequality therefore gives $1 \le 2\varepsilon$.

The identity decoder attains $1/2$, since

$$\Delta_Z = \frac{1}{2}\text{id} + \frac{1}{2}\text{Ad}_Z$$

and $d_{\diamond}(Ad_Z, id) = 1$, witnessed by $|+\rangle$. Global recovery follows from Theorem .

Both marginals in this example depend on the logical populations. Consequently, complete no-hiding does not assert that either marginal must carry the whole qubit. This is the distinction also illustrated by the supplied Em05-A noise comparison [CheungEm05A].

Restricted ensembles can hide information in correlations

Example. Define

$$V|0\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \qquad V|1\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}.$$

The two basis inputs have identical maximally mixed marginals on both outputs, yet orthogonal complete outputs. However,

$$V\frac{|0\rangle + |1\rangle}{\sqrt{2}} = |00\rangle, \qquad V\frac{|0\rangle - |1\rangle}{\sqrt{2}} = |11\rangle.$$

The reduced channels are therefore not constant on the full code. There is no contradiction with Corollary .

Mixed quantum encryption requires a complete purifier

Proposition. Let U_k, $k = 1, \ldots, d^2$, be a Weyl unitary basis. The channel

$$\rho \text{longmapsto} \frac{1}{d^2}\sum_k U_k \rho U_k^\dagger \otimes |k\rangle\langle k|_K$$

has constant marginals on the ciphertext R and key K, but exact recovery from RK. This does not contradict no-hiding.

Proof. The Weyl twirl gives the ciphertext marginal I_R/d; the key marginal is uniformly distributed. Controlled application of $U_k^{\dagger}$ recovers ρ.

A pure complete dilation is

$$V|\psi\rangle = \frac{1}{d}\sum_k U_k|\psi\rangle_R|k\rangle_K|k\rangle_F.$$

The complete complement of R is KF, not K. Its off-diagonal key-purifier matrix entries contain

$$\frac{1}{d^2}\mathrm{Tr}\,(U_k\rho U_\ell^\dagger)\,|k,k\rangle\langle\ell,\ell|.$$

They are generally input dependent. Since the ciphertext channel is constant, Corollary proves exact recoverability from the complete KF output.

These examples repair the unsupported statement that every form of decoherence transfers the entire unknown state into a measurement record or into one environmental marginal.

Quantitative channel certificates

Normalization checks

Set

$$a_d = 1 - d^{-2}, \qquad \tau_C = I_C/d.$$

Lemma. For every code density operator σ,

$$d_\diamond(\mathrm{id}_C, \mathscr{T}_\sigma) \geq a_d, \qquad d_\diamond(\mathrm{id}_C, \mathscr{T}_{\tau_C}) = a_d.$$

Thus a constant accessible channel has optimal recovery error a_d.

Proof. For the Bell input, a replacement output is $(I_Q/d) \otimes \sigma$, whose overlap with $|\Phi_d\rangle$ is $d\^\{-2\}$. The Bell-projector test proves the lower bound.

The Weyl twirl represents

$$\mathscr{T}_{\tau_C} = d^{-2}\sum_k \mathrm{Ad}_{U_k}.$$

Separating the identity term and using that unitary channels have diamond norm one gives the matching upper bound. A decoder after a constant channel remains constant.

Lemma. For isometries V_1, V_2: $C \to R \otimes B$,

$$d_\diamond(\mathscr{V}_1, \mathscr{V}_2) \leq \|V_1 - V_2\|.$$

The same bound holds for either reduced channel.

Proof. For a pure referenced input, the two output vectors differ by at most $\|V_1 - V_2\|$. Their pure-state trace distance is no larger than that vector distance. Convexity handles mixed inputs, and partial trace is contractive.

For a channel Φ, use the normalized Choi state

$$J_n(\Phi) = (\mathrm{id}_Q \otimes \Phi)(|\Phi_d\rangle\langle\Phi_d|).$$

Lemma. For channels Φ, Ψ with input dimension d,

$$d_\diamond(\Phi, \Psi) \leq \min\left\{1, \frac{d}{2}\|J_n(\Phi) - J_n(\Psi)\|_1\right\}.$$

If

$$\|(\Phi - \Psi)(E_{ij})\|_1 \leq r \quad \text{for every} i, j,$$

then

$$d_\diamond(\Phi, \Psi) \leq \min\{1, dr/2\}.$$

Proof. A pure referenced input can be written as

$$|\psi\rangle = (L \otimes I_C)|\Phi_d\rangle, \qquad \mathrm{Tr}\,(L^\dagger L) = d, \qquad \|L\|^2 \leq d.$$

Multiplication on both sides of the Choi difference therefore bounds its trace norm by d times the Choi trace norm.

Alternatively, write

$$|\psi\rangle = \sum_i |v_i\rangle_Q |i\rangle_C, \qquad \sum_i \|v_i\|^2 = 1.$$

Its output difference has trace norm at most

$$r \sum_{i,j} \|v_i\| \|v_j\| \le dr.$$

Divide by two, use convexity, and take the input supremum.

The dimension factor is indispensable to these conversions. One off-diagonal test alone is not enough for a growing code; a full operator basis is required.

The declared external approximate-correction input

For finite-dimensional complementary channels, the KSW information–disturbance theorem gives

$$\varepsilon_{\text{rec}} \le \sqrt{2\delta_{\text{full}}}, \qquad \delta_{\text{full}} \le \sqrt{2\varepsilon_{\text{rec}}}.$$

Indeed its full-norm inequality

$$\frac{1}{4}\Big(\inf_{\mathcal{D}} \|\mathcal{D}\mathcal{N} - \mathrm{id}\|_\diamond\Big)^2 \le \inf_{\sigma} \|\mathcal{C} - \mathcal{T}_\sigma\|_\diamond$$

becomes the first inequality after substituting *2ε_rec* and *2δ_full*. The corresponding converse gives the second [KSW]. We apply these bounds to finite proxies below; no unstated infinite-dimensional attainment assertion is needed.

Theorem. Let $V\colon C \to R \otimes B$ be a finite-dimensional isometry embedded into a common complete output space. Suppose

$$d_\diamond(\mathcal{V}, \widehat{\mathcal{V}}) \le b.$$

For a finite residual density operator $\hat{o}$, put

$$e = \frac{1}{2}\Big\| J_{\text{n}}(\widehat{\mathcal{C}}) - \frac{I_Q}{d} \otimes \hat{\sigma} \Big\|_1.$$

Then

$$\delta_{\text{full}}(\mathcal{C}) \le \min\{1, de + b\}.$$

There exists a finite CPTP decoder, completed on the full radiation space, for which

$$d_\diamond(\mathcal{D}\mathcal{N}, \mathrm{id}_C) \le \min\{1, \sqrt{2de} + b\}.$$

Accordingly,

$$\varepsilon_{\text{rec}}(\mathcal{N}) \le \min\{a_d, \sqrt{2de} + b\}.$$

Independently, if a specified finite CPTP decoder $\mathcal{D}$ has

$$r_{\text{dec}} = \frac{1}{2}\Big\| J_{\text{n}}(\widehat{\mathcal{D}\mathcal{N}}) - |\Phi_d\rangle\langle\Phi_d| \Big\|_1,$$

its complete extension satisfies

$$d_\diamond(\mathcal{D}\mathcal{N}, \mathrm{id}_C) \le \min\{1, dr_{\text{dec}} + b\}.$$

Proof. Partial trace of the complete-channel comparison gives

$$d_\diamond(\mathcal{C}, \widehat{\mathcal{C}}) \le b, \qquad d_\diamond(\mathcal{N}, \widehat{\mathcal{N}}) \le b.$$

The triangle inequality and Lemma prove .

For the finite channel, *δ_full* ≤ *de*. The finite decoder set is compact, so an optimal decoder exists. Equation gives its recovery error at most *√(2de)*.

If *P_R* is the projection onto the embedded *R*, complete this decoder by

$$\mathcal{D}(Y) = \widehat{\mathcal{D}}(P_R Y P_R) + \mathrm{Tr}\,((I - P_R)Y)\tau_C.$$

The map is CP, and its two traces sum to *Tr Y*. Postprocessing contraction adds at most *b*, proving . The constant decoder supplies the additional bound *a_d*.

For the specified decoder, Lemma gives the finite error *dr_dec*. The same extension and comparison prove .

Remark. The regulator allowance *b* already compares complete channels in diamond distance. It is not multiplied by *d*. Moreover, applying finite recovery first and then lifting the decoder gives √*(2de) + b*, rather than putting that comparison allowance inside the square root. Neither expression estimates *e* from dynamics.

A quantitative incompatibility

Proposition. If $\mathcal{N}_rep$ is a replacement channel and

$$d_{\diamond}(\mathcal{N},\mathcal{N}_{\mathrm{rep}}) \le \eta,$$

then

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}) \ge \max\{0, a_d - \eta\}.$$

For finite-dimensional complete outputs,

$$a_d \le \eta + \sqrt{2\delta_{\mathrm{full}}}.$$

Proof. For every decoder, postprocessing contraction gives

$$d_{\diamond}(\mathcal{D}\mathcal{N},\mathrm{id}_C) \ge d_{\diamond}(\mathcal{D}\mathcal{N}_{\mathrm{rep}},\mathrm{id}_C) - \eta.$$

The decoded replacement is constant, so Lemma gives the lower bound. Combine it with .

This proposition concerns a *uniform complete channel* comparison, not agreement with a thermal spectrum on selected observables.

A regulated interacting microscopic application

We now specify the architecture of Hawk02-E [CheungHawk02E] and prove the analytical inputs used here. The proof applies to every frozen coupling array. It does not assume that the array has a useful radiation-recovery regime.

Frozen source and reservoir

Let $N = 2n \ge 4$, with Majoranas

$$\chi_i^{\dagger} = \chi_i, \qquad \{\chi_i,\chi_j\} = 2\delta_{ij} I,$$

acting on $\mathcal{H}_S = \mathcal{F}_f(\mathbb{C}\wedge n)$. Freeze real quartic coefficients and set

$$H_{\mathrm{bare}} = \sum_{i<j<k<l} J_{ijkl}\chi_i\chi_j\chi_k\chi_l,$$

$$B_N = \sum_{i<j<k<l} |J_{ijkl}|,$$

$$H_S = H_{\mathrm{bare}} + B_N I, \qquad M_N = 2B_N.$$

Every quartic monomial is self-adjoint with norm one, so

$$0 \le H_S \le M_N I.$$

Write

$$S_j = \mathrm{i}\chi_{2j-1}\chi_{2j}, \qquad O_{s'} = \frac{1}{n}\sum_{j=1}^{n} s_j' S_j, \qquad A = \alpha O_{s'} + \zeta O_{s'}^2.$$

The S_j commute, and $\|A\| \le |\alpha| + |\zeta|$. For the estimates, one may use a fixed finite family of even self-adjoint A_a, including the choice $A_a = A$.

Fix

$$0 < \mu < \Lambda < \infty, \qquad \mathfrak{h} = L^2([\mu,\Lambda], \mathrm{d}\omega; \mathbb{C}^r), \qquad h_B f(\omega) = \omega f(\omega).$$

On $\mathcal{H}_bath = \mathcal{F}_s(\mathfrak{h})$, let

$$H_B = \mathrm{d}\Gamma(h_B), \qquad f_a(\omega) = \frac{\mathbf{e}_a}{\sqrt{2\omega}}, \qquad \Phi(f) = a(f) + a^\dagger(f),$$

with annihilation conjugate-linear in *f*. The inner product is conjugate-linear in its first argument.

For frozen real *g_a*, define

$$H = H_S + H_B + V_{\mathrm{int}}, \qquad V_{\mathrm{int}} = \sum_a g_a A_a \otimes \Phi(f_a).$$

Set

$$G = \sum_a |g_a| \|A_a\| \|f_a\|,$$

$$G_\omega = \sum_a |g_a| \|A_a\| \left\| h_B^{-1/2} f_a \right\|.$$

Here

$$\|f_a\|^2 = \frac{1}{2} \log \frac{\Lambda}{\mu}, \qquad \left\| h_B^{-1/2} f_a \right\|^2 = \frac{1}{2}\left(\frac{1}{\mu} - \frac{1}{\Lambda}\right).$$

This is a fixed-realization quartic SYK–scalar-reservoir specification. The normalization is printed rather than inferred from a different SYK convention. The bath is initially vacuum. No gravitational geometry is assigned to these objects.

Existence of complete unitary dynamics

Theorem. The Hamiltonian is self-adjoint on $\mathscr{H}_S \otimes D(H_B)$, bounded below, and satisfies the form inequality

$$H \geq \frac{1}{2} H_B - 2G_\omega^2 I.$$

It generates *U(t) = e^{−itH}*. For every isometric diary loading and fixed pure reservoir preparation, the complete diary channel at every finite time is exactly recoverable.

Proof. The annihilation estimate is

$$\|a(f)\psi\| \leq \left\| h_B^{-1/2} f \right\| \left\| H_B^{1/2} \psi \right\|.$$

The CCR give

$$\left\| a^\dagger(f)\psi \right\|^2 = \|a(f)\psi\|^2 + \|f\|^2 \|\psi\|^2.$$

Therefore

$$\|V_{\mathrm{int}}\psi\| \leq 2G_\omega \left\| H_B^{1/2} \psi \right\| + G\|\psi\|.$$

For every *b > 0*, the spectral theorem bounds the square-root term by *b‖H_Bψ‖ + c_b‖ψ‖*. Thus the interaction has relative operator bound zero. Kato–Rellich proves self-adjointness on the stated domain; the source term is bounded. These are the standard field estimates used in spin–boson Hamiltonian constructions [HaslerHerbst].

For normalized vectors in the form domain,

$$|\langle V_{\mathrm{int}} \rangle| \leq 2G_\omega \sqrt{\langle H_B \rangle} \leq \frac{1}{2}\langle H_B \rangle + 2G_\omega^2.$$

Since *H_S ≥ 0*, the lower bound follows. The spectral theorem supplies the unitary group. Apply Theorem to the complete prepared evolution.

This is an actual preservation theorem for the printed microscopic Hamiltonian, not a proof that quantum gravity uses that Hamiltonian.

A coherent diary and its displaced carrier

Choose a simultaneous spin eigenvector $|B_s\rangle$ of fixed source parity, and prepare

$$|\Omega_S\rangle = Z_s(\beta)^{-1/2} e^{-\beta H_S/2} |B_s\rangle, \qquad Z_s(\beta) = \langle B_s \mathrm{rverte}^{-\beta H_S} |B_s\rangle > 0.$$

This Euclidean-filtered preparation is motivated by Kourkoulou and Maldacena [KourkoulouMaldacena]. The exact normalization here follows from *e^{−βM_N} ≤ Z_s(β) ≤ 1*.

The source preparation is completed before the diary arrives. If it is heralded, its success decision is independent of the diary. An attempted deterministic preparation protocol must retain failure outputs in its own complete channel.

Encode a qubit in two carrier modes:

$$J_D|0\rangle = |00\rangle_D, \qquad J_D|1\rangle = |11\rangle_D.$$

Both basis vectors are even. Choose two source modes *T*, fix a fermionic occupation ordering, and let *U_sw* be the second quantization of their one-particle exchange with the carrier. Define

$$W|\psi\rangle = U_{\mathrm{sw}}(J_D|\psi\rangle \otimes |\Omega_S\rangle).$$

This is one message-independent, parity-preserving isometry.

The even diary introduces no logical-input-dependent permutation sign. The old source two-mode state is displaced into the carrier, including its correlations with the remaining source. The carrier marginal is diary independent, but it is generally neither vacuum nor uncorrelated with the source. It remains in the complete residual. This differs from the vacuum destination used in Hawk02-D [CheungHawk02D].

Fix an orthonormal packet family *p_1*, …, *p_q* and

$$\mathfrak{r} = \mathrm{span}\,\{p_1, \ldots, p_q\}.$$

The declared access partition is

$$R = \mathscr{F}_{\mathrm{s}}(\mathfrak{r}), \qquad B = \mathscr{H}_S \otimes \mathscr{H}_D \otimes \mathscr{F}_{\mathrm{s}}(\mathfrak{r}^{\perp}).$$

After the fixed regrouping,

$$V_t|\psi\rangle = \mathcal{R}\,(U(t) \otimes I_D)(W|\psi\rangle \otimes |\Omega_{\mathrm{bath}}\rangle).$$

Its complete matrix units are

$$\mathscr{N}_t(E_{ij}) = \mathrm{Tr}_B\,|V_t i\rangle\langle V_t j|, \qquad \mathscr{C}_t(E_{ij}) = \mathrm{Tr}_R\,|V_t i\rangle\langle V_t j|.$$

Energy accounting

Proposition. For every logical input, the prepared evolution obeys

$$E_S(t) + E_B(t) + E_I(t) = E_0, \qquad E_0 = \mathrm{Tr}\,(H_S W \rho W^{\dagger}) \in [0, M_N],$$

where *E_I* = ⟨*V_int*⟩. Moreover,

$$0 \le E_B(t) \le 2M_N + 4G_{\omega}^2.$$

On the prepared finite-time domain,

$$-\dot{E}_S = -\mathrm{i}\langle[V_{\mathrm{int}}, H_S]\rangle, \qquad \dot{E}_B = -\dot{E}_S - \dot{E}_I.$$

Proof. The initial bath and interaction means vanish. Autonomous unitary evolution conserves the total Hamiltonian expectation. The form bound in Theorem gives the bath-energy bound.

The occupation estimates in the next subsection give finite moments of every fixed particle-number order on bounded time intervals. Since the frequency band is bounded, differentiation can be justified by finite-particle approximation. The commutator identity and differentiated conservation law then give the currents.

Neither positive bath energy nor vanishing source energy establishes a small complete residual. Interaction energy cannot be omitted or renamed as black-hole mass.

Finite approximation of the complete dilation

Let *P_m* project onto total bath occupation at most *m*. For *x* ≥ *0*, put

$$Z_m(x) = \sum_{\ell = m+1}^{\infty} \frac{x^{\ell}}{\sqrt{\ell!}}.$$

Theorem. At fixed *μ*, *Λ*, every finite observation interval *[0, T]* and finite packet family admit a finite-dimensional prepared isometry *V_t* such that

$$d_{\diamond}(\mathscr{V}_t, \widehat{\mathscr{V}}_t) \le b_{\mathrm{reg}}(T; m, \Delta) := \min\left\{1, 2Z_m(2GT) + \frac{Tm\Delta}{2}\right\}$$

uniformly for $t \in [0, T]$. The same bound holds for both complete complementary outputs after their common embeddings.

For $0 < b < 1$, it suffices to choose $m \ge 1$ and a mesh with

$$m + 1 \ge 4e(2GT)^2, \qquad m \ge 2 + \log_2(1/b), \qquad \Delta \le \frac{b}{Tm},$$

when $T > 0$. Then $b_reg \le b$.

Proof. In interaction picture,

$$V_I(s) = \sum_a g_a A_a(s) \otimes \Phi(e^{\mathrm{ish}_B} f_a).$$

On sectors containing at most k particles,

$$\|V_I(s)\psi\| \le 2G\sqrt{k+1}\,\|\psi\|.$$

An ℓ-fold product acting on bath vacuum has norm at most $(2G)^{\ell}\sqrt{(\ell!)}$. The ordered integration simplex has volume $t^{\ell}/\ell!$. The vacuum Dyson term is consequently bounded by

$$\frac{(2Gt)^{\ell}}{\sqrt{\ell!}}.$$

The bound is uniform over source, carrier, and reference inputs.

The series converges, also after multiplication by any fixed power of ℓ. Since each term has occupation at most ℓ, it converges in every fixed particle-number graph norm. On these vectors the differentiated series satisfies the Schrödinger equation; uniqueness for the self-adjoint Hamiltonian identifies it with the actual evolution.

The evolution generated by P_mHP_m, on its projected space, has the same Dyson terms through order m. Inserted projections are contractions, so the difference of the two prepared isometries is at most $2Z_m(2GT)$.

Partition $[\mu, \Lambda]$ into intervals of width at most Δ, and replace frequency multiplication by its bin midpoint value $\hbar_B$. Then

$$\left\|h_B - \widetilde{h}_B\right\| \le \Delta/2, \qquad \left\|P_m \mathrm{d}\Gamma(h_B - \widetilde{h}_B) P_m\right\| \le m\Delta/2.$$

Duhamel's formula adds at most $Tm\Delta/2$.

For an exact finite active-mode realization of this discretized model, take

$$\mathfrak{f} = \mathrm{span}\{1_{I_b} f_a, 1_{I_b} p_j\}.$$

It contains all coupling vectors and the complete packet space, and is invariant under $\hbar_B$. Its orthogonal complement remains vacuum. Thus the occupation-truncated dynamics on $\mathscr{F}_s(\mathfrak{f})$ is finite dimensional.

The total-occupation subspace is embedded into the product

$$\mathscr{F}_{\mathrm{s_}} \le m(\mathfrak{r}) \otimes \mathscr{F}_{\mathrm{s_}} \le m(\mathfrak{f} \cap \mathfrak{r}^{\perp}).$$

This is an embedding, not an assertion that the total cutoff itself factorizes. The source and carrier are retained. Apply Lemma and partial trace.

For the explicit selection, let $x = 2GT$. The first condition gives

$$\frac{x^{m+1}}{\sqrt{(m+1)!}} \le \left(\frac{ex^2}{m+1}\right)^{(m+1)/2} \le 2^{-(m+1)}.$$

The subsequent tail-term ratios are at most $1/2$, so $Z_m(x) \le 2^{-m}$. The second condition makes $2Z_m(x) \le b/2$; the mesh condition bounds the remaining term by $b/2$.

The theorem refines finite occupation and frequency proxies. It does *not* remove the fixed infrared or ultraviolet cutoffs. The constants displayed above are not uniform as $\mu \downarrow 0$ or $\Lambda \uparrow \infty$.

An exact dynamical negative control

Commuting source couplings

Theorem. If

$$[H_S, A_a] = 0, \qquad [A_a, A_b] = 0$$

for all a, b, the reservoir channel from any inserted finite code is entanglement breaking. Every deterministic radiation decoder has error at least

$$1 - d^{-1}.$$

Proof. Choose common spectral projectors P_{λ} of the source operators. The propagator is block diagonal:

$$U(t) = \sum_{\lambda} P_{\lambda} \otimes e^{-\mathrm{i}t(E_{\lambda} + H_B + \sum_a g_a a_{a,\lambda} \Phi(f_a))}.$$

Tracing the source kills cross-projector terms. The accessible channel therefore has the form

$$\mathcal{N}_t(\rho) = \sum_{\lambda} \mathrm{Tr}\,(F_{\lambda}\rho)\,\sigma_{\lambda,t}, \qquad F_{\lambda} = W^{\dagger}(P_{\lambda} \otimes I_D)W.$$

The F_{λ} form a POVM. Thus the channel is measure-and-prepare.

A decoder preserves separability of its Bell-input output. Every separable state on $Q \otimes C$ has Bell overlap at most $1/d$, by Cauchy–Schwarz for product vectors and convexity. The Bell-projector test proves the lower bound.

Corollary. If $\mathcal{N}_{cl}$ is entanglement breaking and $d_{\diamond}(\mathcal{N}, \mathcal{N}_{cl}) \le \eta$, then

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}) \ge \max\{0, 1 - d^{-1} - \eta\}.$$

Proof. Use postprocessing contraction and Theorem 's separable Bell-state argument.

This repairs the proposed shortcut of replacing source operators by classical histories while assuming that quantum recovery survives. A classical-history approximation needs its own complete-channel error; if that error is small, it gives an obstruction rather than a high-accuracy decoder.

One frozen four-Majorana realization

We can evaluate a member of the interacting family exactly, without a disorder average or an approximate scrambling premise.

Fix

$$N = 4, \qquad J_{1234} = J > 0, \qquad \alpha = 1, \quad \zeta = 0, \quad s' = (1, 1),$$

one bath species, $g \neq 0$, and any fixed $0 < \mu < \Lambda$. Then

$$H_S = J(I - S_1 S_2), \qquad A = \frac{S_1 + S_2}{2}.$$

Take $c_j = (\chi_{2j-1} + i\chi_{2j})/2$, so $S_j = 2c_j^{\dagger}c_j - I$.

Prepare $|B_s\rangle = |11\rangle_S$. Its filtered state is still $|11\rangle_S$. Swap the entire two-mode diary carrier with the source. The carrier becomes the fixed state $|11\rangle_D$, and the logical source states are $|00\rangle_S$, $|11\rangle_S$. Both have source energy zero, and their coupling eigenvalues are -1, $+1$, respectively.

For

$$f(\omega) = \frac{1}{\sqrt{2\omega}},$$

define

$$\eta_i(t) = g a_i \frac{e^{-\mathrm{i}\omega t} - 1}{\omega} f(\omega), \qquad a_0 = -1, \quad a_1 = 1.$$

These are square-integrable coherent amplitudes. For the declared packet projection $P_{\mathfrak{r}}$, put

$$\eta_{i,R}(t) = P_{\mathfrak{r}}\eta_i(t), \qquad c_R(t) = \exp\left[-\frac{1}{2}\|\eta_{0,R}(t) - \eta_{1,R}(t)\|^2\right],$$

and

$$z_R(t) = \sqrt{1 - c_R(t)^2}.$$

Theorem. For this realization,

$$\mathcal{N}_t(\rho) = \rho_{00}|\eta_{0,R}(t)\rangle\langle\eta_{0,R}(t)| + \rho_{11}|\eta_{1,R}(t)\rangle\langle\eta_{1,R}(t)|.$$

Its optimal deterministic radiation recovery error is

$$\text{boxed}\,\varepsilon_{\text{rec}}(t) = \frac{3 - z_R(t)}{4} \geq \frac{1}{2}.$$

The complete residual obeys *δ_full(t) ≥ 1/2* at every time. The complete output nevertheless preserves the qubit exactly.

If all bath modes are accessible, then

$$c_R(t) = \exp\left[-2g^2\int_{\mu}^{\Lambda}\frac{1 - \cos(\omega t)}{\omega^3}\mathrm{d}\omega\right].$$

These statements are finite-realization dynamical conclusions, not properties inferred from source size or chaos.

Proof. On source eigenvalue *a_i*, the bath Hamiltonian is *H_B + ga_iΦ(f)*. Let *q_i = ga_ih_B^{−1}f*, and use the Weyl unitary

$$\mathcal{W}(q) = e^{a^\dagger(q) - a(q)}.$$

The CCR give

$$H_B + ga_i\Phi(f) = \mathcal{W}(q_i)^\dagger H_B \mathcal{W}(q_i) - g^2a_i^2\left\|h_B^{-1/2}f\right\|^2 I.$$

Thus vacuum evolves, up to a phase, to the coherent state with amplitude *(e^{−ith_B} − I)q_i = η_i(t)*. The phase is common for *a_i = ±1*. Coherent states factor under the orthogonal packet decomposition. Tracing the orthogonal source states gives .

For two pure radiation states with positive overlap *c_R*, their trace distance is *z_R = √(1 − c_R^2)*. The optimal equal-prior discrimination success probability is *(1 + z_R)/2*. This follows directly by maximizing

$$\frac{1}{2}[\,\mathrm{Tr}\,(E\sigma_0) + \mathrm{Tr}\,((I - E)\sigma_1)\,]$$

over effects *0 ≤ E ≤ I*, using the positive part of *σ_0 − σ_1*.

For any decoder, the decoded Bell overlap is

$$F_e = \frac{1}{4}[\,\langle 0|\mathcal{D}(\sigma_0)|0\rangle + \langle 1|\mathcal{D}(\sigma_1)|1\rangle\,].$$

The two diagonal decoder effects form a discrimination POVM. Hence *F_e ≤ (1 + z_R)/4*, and the Bell-projector test gives error at least *(3 − z_R)/4*.

The two states are interchanged by radiation parity. Their Helstrom measurement can therefore be chosen with symmetric conditional errors. Measuring it and preparing *|0⟩* or *|1⟩* produces the complete decoded channel

$$z_R\Delta_Z + (1 - z_R)\mathcal{T}_{I_2/2}.$$

Its Pauli probabilities are

$$p_I = p_Z = \frac{1 + z_R}{4}, \qquad p_X = p_Y = \frac{1 - z_R}{4}.$$

Its distance from the identity is *1 − p_I = (3 − z_R)/4*: the Bell-state outputs are orthogonal for the four Paulis, and the triangle inequality gives the same upper bound. The decoder is completed on the unused radiation subspace by fixed-state preparation. At *z_R = 0*, use the constant decoder.

The complete residual retains orthogonal source states on the two logical basis inputs. Thus their residual trace distance is one. If both residual outputs were within distance *δ* of one replacement state, the triangle inequality would give *1 ≤ 2δ*. This proves *δ_full ≥ 1/2*.

Global preservation follows from Theorem . Finally,

$$\|\eta_0 - \eta_1\|^2 = 4g^2 \int_\mu^\Lambda \frac{1 - \cos(\omega t)}{\omega^3} \mathrm{d}\omega,$$

which gives the full-bath overlap.

Remark. This nonzero-coupling example belongs to the printed quartic family, but its operators commute. It rules out a recovery claim for *every* frozen realization or parameter choice. It does not rule out recovery for larger noncommuting realizations. No such positive realization is certified in this paper.

Proposition. For the realization above,

$$E_S(t) = 0, \qquad E_B(t) = g^2 \int_\mu^\Lambda \frac{1 - \cos(\omega t)}{\omega^2} \mathrm{d}\omega, \qquad E_I(t) = -E_B(t).$$

Positive bath energy therefore coexists with no source-energy loss and with the recovery obstruction .

Proof. The source remains in its even zero-energy eigenspace. The coherent bath energy is ‖h_B^{1/2}η_i(t)‖^2, giving the integral. The initially prepared total energy is zero, and conservation gives the interaction term.

This energy comes from relaxation of the initially undressed interaction preparation. It is not a derived gravitational evaporation law.

What a positive finite-time certificate must contain

Theorem makes the following procedure mathematically well posed for another frozen realization:

• freeze the code, preparation, array, couplings, and packet algebra;

• select T, b, m, Δ before computing a certificate;

• compute all complementary logical matrix units of the finite prepared isometry;

• certify a residual state or a complete decoder;

• apply Theorem . A code dimension d is fixed before the Choi conversion.

A finite residual optimization

For the finite complementary Choi state J, a trace-norm certificate can be obtained from

$$\begin{aligned} &\text{minimize} && \tfrac{1}{2}\mathrm{Tr}\,(X_+ + X_-), \\ &\text{subject to} && J - \tfrac{I_Q}{d} \otimes \sigma = X_+ - X_-, \\ & && X_\pm \succeq 0, \quad \sigma \succeq 0, \quad \mathrm{Tr}\,\sigma = 1. \end{aligned}$$

Every rigorously feasible tuple gives an upper bound on e. The objective is a Choi trace distance, not the exact diamond norm.

A decoder search uses an unnormalized decoder Choi operator satisfying

$$J_D \succeq 0, \qquad \mathrm{Tr}_C\, J_D = I_R^{\wedge}.$$

Composition with the known finite channel is linear. A feasible decoder and a validated bound on its decoded Choi distance supply .

An exploratory floating-point optimum is not a proof of positivity, trace preservation, or the claimed error. No numerical execution is asserted here.

One certified point can give a finite interval

Proposition. Let $\mathscr{D}$ be a fixed finite CPTP decoder. For the finite proxy, define

$$L_m = M_N + m\Lambda + 2G\sqrt{m+1}.$$

Suppose a validated computation at $t_0 \in (0, T)$ proves $r_{dec}(t_0) \le r_0$. Then for the same decoder,

$$d_\diamond(\mathscr{D}\mathscr{N}_t, \mathrm{id}_C) \le dr_0 + dL_m|t - t_0| + b_{\mathrm{reg}}$$

throughout $[0, T]$, capped at one.

If

$$dr_0 + b_{\mathrm{reg}} < \varepsilon,$$

every

$$|t - t_0| < \frac{\varepsilon - dr_0 - b_{\mathrm{reg}}}{dL_m}$$

inside *[0, T]* belongs to a nonempty certified recovery interval.

Proof. The finite Hamiltonian has operator norm at most *L_m*. Therefore its prepared isometries differ by at most *L_m|t − t_0|*. Lemma , decoder contraction, and the Bell-input test give

$$r_{\mathrm{dec}}(t) \le r_0 + L_m|t - t_0|.$$

Apply .

This proposition certifies an interval *if* its strict pointwise premise is actually proved. It does not furnish that premise for the interacting model.

Disorder is not averaged away

Proposition. For a qubit, define

$$\mathscr{C}_0(\rho) = \mathrm{diag}\,(\rho), \qquad \mathscr{C}_1(\rho) = X\mathrm{diag}\,(\rho)X.$$

Their equal average is the replacement channel *Tr (ρ)I/2*, although each realization perfectly distinguishes the logical basis inputs.

Proof. Direct addition gives the average. Each individual pair of basis outputs is orthogonal.

If a declared probability space instead proves

$$\mathbb{E}_{\mathbf{J}} e_{\mathbf{J}}(t) \le a(t),$$

Markov's inequality gives, with probability at least *1 − p*,

$$\delta_{\mathrm{full}}_\mathbf{J}(t) \le \min\{1, d\,a(t)/p + b_{\mathbf{J}}\}.$$

The expectation here is an expectation of distances, not the distance of an averaged channel.
Realization-dependent decoders and any necessary realization knowledge must be declared. A uniform-in-time probability statement requires additional control.

Compatible errors only

If preparation, collection, and implementation have proved half-diamond allowances at named interfaces, then a specified decoder obeys

$$\varepsilon_{\mathrm{tot}} \le dr_{\mathrm{dec}} + b_{\mathrm{reg}} + b_{\mathrm{prep}} + b_{\mathrm{collection}} + b_{\mathrm{implementation}},$$

capped at one. The proof is telescoping CPTP composition and contraction. This is the discipline used in the supplied En02-B and Em03-C comparisons [CheungEn02B,CheungEm03C].

Uncollected weight already included in the complete residual is not charged again as a separate loss. A metric error is not a diamond error without a proved conversion.

Endpoints and the missing gravitational implications

Access-preserving output identifications

Definition. For channels with time-dependent output partitions *R_t ⊗ B_t*, an access-preserving comparison consists of isometries

$$J_{R,t} : R_t \to R_\infty, \qquad J_{B,t} : B_t \to B_\infty.$$

The embedded complete isometry is

$$\overline{V}_t = (J_{R,t} \otimes J_{B,t})V_t.$$

These maps identify outputs; they do not transfer an inaccessible logical subsystem into the radiation access.

Lemma. The leakage distance of $\mathscr{C}_t$ is unchanged under the residual isometric embedding $J_{B, t}$.

Proof. Embedding any replacement state proves one inequality. For the other, choose a fixed state τ on B_t and use the CPTP retraction

$$Y \longmapsto J_{B,t}^{\dagger} Y J_{B,t} + \operatorname{Tr}\left((I - J_{B,t} J_{B,t}^{\dagger}) Y\right) \tau.$$

It exactly reverses embedded outputs. Contraction applied to any candidate replacement state proves the reverse inequality.

Theorem. Suppose

$$\overline{V}_t |i\rangle \to V_\infty |i\rangle$$

for every code basis vector under access-preserving identifications. Then convergence is in operator norm, V_{∞} is an isometry, and, with

$$a_t = \left\| \overline{V}_t - V_\infty \right\|,$$

$$|\delta_{\text{full}}(t) - \delta_\infty| \le a_t.$$

Thus a proved limit $\delta_{full}(t) \to 0$ implies $\mathscr{C}_{\infty} = \mathscr{T}_{\sigma}$ and exact endpoint radiation recovery.

Proof. For a finite code,

$$\left\| \overline{V}_t - V_\infty \right\|^2 \le \sum_i \left\| (\overline{V}_t - V_\infty) |i\rangle \right\|^2 \to 0.$$

Inner products pass to the limit, proving the isometry property. Lemma bounds the complementary channel difference by a_t. Distance to a fixed set of replacement channels is one-Lipschitz. Lemma identifies the finite-time leakage.

If $\delta_{\infty} = 0$, choose replacement states σ_k with channel distance tending to zero. Evaluation on τ_C gives $\sigma_k \to \mathscr{C}_{\infty}(\tau_C)$ in trace distance. The replacement channels converge as well, so $\mathscr{C}_{\infty}$ is exactly constant. Apply Theorem .

Remark. A general output unitary S_t that mixes radiation and residual does not preserve the original recovery task. For example,

$$V|\psi\rangle = |0\rangle_R |\psi\rangle_B$$

has leakage a_d, whereas a swap of R and B produces zero leakage in the new partition.

A derived scattering representation may define a legitimate new asymptotic access algebra, as in Hawk02-D [CheungHawk02D]. Its relation to finite-time access must nevertheless be proved. Strong convergence alone does not supply that relation.

Finite proxies do not prove permanent release

Proposition. At fixed finite occupation cutoff and frequency mesh, the autonomous proxy has arbitrarily late times at which its prepared channel returns arbitrarily closely to its initial channel. The initially vacuum radiation channel is constant. Consequently,

$$\limsup_{t \to \infty} \varepsilon_{\text{rec}}(\widehat{\mathscr{N}_t}) = a_d.$$

Proof. A finite-dimensional Hamiltonian has finitely many eigenvalues. Simultaneous recurrence of their phases gives arbitrarily large times at which the unitary is arbitrarily close to the identity. Lemma then gives channel recurrence.

Optimal recovery error is one-Lipschitz in channel diamond distance: insert any decoder and use contraction, then take infima in both directions. The initial error is a_d. For every channel, the constant decoder has error a_d, giving the matching upper bound.

The finite-time continuum approximation must therefore precede any controlled outgoing limit. It is not legitimate to infer an endpoint by extrapolating a fixed finite proxy.

Entropy and source energy are insufficient

The two isometries

$$V_{\text{good}}|\psi\rangle = |\psi\rangle_R|0\rangle_B, \qquad V_{\text{bad}}|\psi\rangle = |0\rangle_R|\psi\rangle_B$$

give radiation entropy zero for every pure code input. Their optimal radiation recovery errors are respectively *0* and *a_d*. Thus a selected radiation entropy does not characterize the channel.

Likewise, *V_bad* can have a source in a fixed zero-energy vacuum while an inaccessible degenerate memory retains the code. Vanishing source energy is not complete residual independence. These are the same logical distinctions emphasized in Hawk01-B and Hawk02-E [CheungHawk01B,CheungHawk02E].

The conditional black-hole conclusion

Theorem. Suppose a gravitational model supplies:

• a justified finite code and complete microscopic dilation;

• unitary complete evolution with the preparation ledger retained;

• a derived outgoing access and endpoint construction;

• a proof that the endpoint complete residual channel is constant. Then the outgoing radiation exactly recovers the code, including arbitrary reference entanglement.

If instead all nonradiative output degrees of freedom are proved to be a single fixed pure state, the complete radiation channel is an isometric encoding. A constant final radiation channel on a nontrivial code is incompatible with these hypotheses.

Proof. The complete microscopic evolution supplies an isometry by Theorem . Apply Theorem to the derived endpoint partition. The rank-one case follows from Proposition . A constant radiation channel cannot have a CPTP left inverse, by Lemma .

The theorem does not establish its gravitational hypotheses. In particular:

• black-hole unitarity is not proved by assuming it;

• a horizonless saddle is not a proof that every inaccessible microscopic degree of freedom has become diary independent;

• a single-run quantum decoder is not reconstruction of a classical preparation label through repeated experiments;

• absence of a remnant geometry is not, without a dictionary, absence of a residual information sector.

The scalar-bath model of Gaikwad, Kaushal, Mandal, and Wadia has explicit low-energy evaporation solutions and a repeated-process initial-label reconstruction scheme [GaikwadEtAl]. Its bath preparation and low-energy assumptions are not identified with the vacuum, finite-band, reference-entangled diary specification here. Its geometry is not attached to our decoder by citation.

Island and replica-wormhole calculations provide substantial entropy and reconstruction results in their stated gravitational settings [AlmheiriEtAl2020,PSSY]. They are not dismissed as scalar curve fits. Conversely, they are not substituted for the missing fixed-realization complete-channel calculation of this manuscript.

For recovery *during* controlled evaporation, the required set is

$$I_{\text{success}} = I_{\text{geometry}} \cap I_{\text{access}} \cap I_{\text{recovery}}.$$

Each interval must be established in the same model. A gravitational approximation allowance remains separate from the channel budget unless a conversion theorem connects them.

Release and conclusion

Theorem. The following statements are proved:

• complete unitary evolution preserves a finite code globally, with an explicit CPTP left inverse and all references retained;

• exact recovery from one output is equivalent to a constant complete complementary channel, with the decoder ;

• exact no-hiding applies to full-channel bleaching, not arbitrary partial decoherence or incomplete environmental

descriptions;

• finite Choi and decoder certificates have the explicit dimension factors and complete-channel allowances of Theorem ;

• the regulated quartic SYK–scalar Hamiltonian has well-defined complete unitary dynamics and the finite-time regulator ;

• one frozen four-Majorana, nonzero-coupling realization has the exact radiation-recovery obstruction , despite exact global preservation;

• a positive finite certificate can extend to a finite interval under Proposition , but no such positive interacting certificate is reported here;

• endpoint continuity requires the stated access-preserving identifications, and fixed finite proxies recur. The gravitational implication is precisely the conditional theorem , not an established physical evaporation solution.

Proof. The statements follow from the named theorems and propositions. Every microscopic object used by those proofs is specified in this source. The physical conclusion requires additional objects absent from the microscopic construction, as listed in Theorem .

Answer to the motivating question. Under complete unitary evolution, an unknown quantum state is not annihilated: the complete channel has a reference-preserving inverse on its encoded range. That is the proved mathematical principle.

The black-hole information problem asks whether a correct physical evaporation model supplies such a complete evolution and whether its outgoing radiation is the output from which the code can be recovered. Those are not consequences of the phrase ``unitary evolution" alone.

Submission and evidence boundary. This is a self-contained analytical manuscript source. A successful pdflatex build, independently executed numerical certificate, formal proof check, independent technical review, and journal acceptance are not asserted. The author should confirm metadata and archival availability of unpublished comparisons before submission. The familiar preservation and no-hiding principles are attributed to their established literature; no priority claim is made.

appendix

A benchmark that does not transfer automatically to SYK

Proposition. Let $A: C \to R$ and $L: C \to B$ be isometries into occupied subspaces orthogonal to their respective vacuum flags. For $0 \le s \le 1$, define

$$V_s|\psi\rangle = \sqrt{1-s}\, A|\psi\rangle|0\rangle_B + \sqrt{s}\, |0\rangle_R L|\psi\rangle.$$

Then

$$\varepsilon_{\mathrm{rec}}(\mathcal{N}_s) = \delta_{\mathrm{full}}(\mathcal{C}_s) = s(1-d^{-2}).$$

Proof. The accessible channel is

$$\mathcal{N}_s(X) = (1-s)AXA^\dagger + s\mathrm{Tr}\,(X)|0\rangle\langle 0|.$$

Decode the occupied block exactly and prepare I_C/d on the vacuum block. The decoded channel is $(1-s)id + s\mathcal{T}_{\tau_C}$, with error sa_d.

For any decoder, the vacuum branch of a Bell input has Bell overlap d^{-2}; the occupied branch has overlap at most one. The Bell-projector test gives the matching lower bound.

The complementary channel is

$$\mathcal{C}_s(X) = sLXL^\dagger + (1-s)\mathrm{Tr}\,(X)|0\rangle\langle 0|.$$

Choosing

$$\sigma = sL\tau_C L^\dagger + (1-s)|0\rangle\langle 0|$$

gives leakage at most sa_d.

For any other replacement state, project onto the occupied code block and vacuum, sending unused output to vacuum. Let *a* be its occupied weight. For the Bell input, the effect equal to the occupied Bell projector and zero on vacuum has expectation difference *s − a/d^2*. The effect equal to that projector on the occupied block and identity on vacuum has difference *a(1 − d^{−2})*. The first is at least *sa_d* if *a ≤ s*; the second is at least *sa_d* if *a ≥ s*. Thus every replacement channel has distance at least *sa_d*.

The supplied Hawk02-D derives this structure from its quadratic fermionic Hamiltonian, with

$$s = s_M(t) = q(t) + \ell_M(t), \qquad q(t) = \left| \frac{J_1(2ct)}{ct} \right|^2,$$

and gives the separate many-particle bound

$$\min \left\{ 1, \sqrt{2[1 - (1 - s_M(t))^{K/2}]} \right\}$$

[CheungHawk02D]. Those are comparison results for that Hamiltonian. A quadratic reduction of another model must be proved before these specific amplitudes can be used. The quartic source coupled to scalar fields need not preserve a one-particle sector, and no such reduction is established here.

Additional debugging checks

A coherent cutoff is not priced by discarded probability

For

$$|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}, \qquad \sigma = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|\perp\rangle\langle\perp|,$$

direct diagonalization in the common three-dimensional space gives

$$d_{\mathrm{tr}}(|+\rangle\langle+|, \sigma) = \frac{\sqrt{5}+1}{4} > \frac{1}{2}.$$

The two-dimensional difference block has eigenvalues *(1 + √(5))/4* and *(1 − √(5))/4*, and the failure block has eigenvalue *−1/2*.

The discarded-probability equality in the supplied En02-E concerns its commuting state-cutoff hypotheses [CheungEn02E]. Our regulator allowance instead follows from the complete isometries.

The Gaussian covariance sign convention

With annihilation conjugate-linear in *f*,

$$e^{itH_B}\Phi(f)e^{-itH_B} = \Phi(e^{ith_B}f),$$

and

$$\langle \Omega_{\mathrm{bath}}, \Phi(e^{ish_B}f_a)\Phi(e^{iuh_B}f_b)\Omega_{\mathrm{bath}}\rangle = \delta_{ab}\int_{\mu}^{\Lambda} \frac{e^{-i\omega(s-u)}}{2\omega}\mathrm{d}\omega.$$

This covariance convention is consistent with the coherent amplitude used in Theorem . A prescribed scalar-history calculation does not replace the noncommuting source channel on all matrix units.

Separate programmes are not discharged

No mathematical statement in this paper is a construction of the characteristic quantum coefficient discussed in the supplied XIV-I/XIV-J excerpts [CheungXIVI,CheungXIVJ]. No interacting Ward identity, coefficient admission, or gravitational instrument is inferred from a finite code isometry.

Under the supplied documentary terminology:

O.1 remains NOTCLOSED; XV-B is not opened; XVI-N is not discharged.

These are scope statements, not new proofs about those programmes. The external catalogue families are not used as theorem premises. A spectral gap, area law, classical geometric inequality, or entropy inequality does not become a complete-residual dynamical estimate merely by being cited.

Minimal reproducibility package

A positive interacting recovery claim should archive:

• the frozen source array, source normalization, code, preparation, carrier ordering, bath cutoffs, couplings, and packet functions;

• the regulator choice made before certificate evaluation;

• the exact finite Hamiltonian specification and controlled propagator evaluation;

• all logical matrix-unit outputs of both channels;

• a rigorously feasible residual state or complete decoder;

• validated positivity, trace-preservation, and trace-norm bounds;

• arithmetic, collection, and implementation allowances at named interfaces;

• the observation time or a proved uniform finite interval;

• for a gravitational claim, the separate code, radiation, backreaction, geometry, and common-interval correspondence;

• for an endpoint claim, the complete outgoing identification, bound/dark-sector audit, residual independence, and order of limits. No item is replaced by a successful branch, population-only test, disorder-averaged channel, or selected entropy history.

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